The brachistochrone problem was posed by Johann Bernoulli in Acta Eruditorum in June 1696. He introduced the problem as follows:

*Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.*

Galileo in 1638 had studied the problem. His version of the problem was first to find the straight line from a point A to the point on a vertical line which it would reach the quickest. He correctly calculated that such a line from A to the vertical line would be at an angle of 45 reaching the required vertical line at B say.

He calculated the time taken for the point to move from A to B in a straight line, then he showed that the point would reach B more quickly if it travelled along the two line segments AC followed by CB where C is a point on an arc of a circle.

Although Galileo was perfectly correct in this, he then made an error when he next argued that the path of quickest descent from A to B would be an arc of a circle - an incorrect deduction.
Cycloid:
A cycloid is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line. It is an example of a roulette, a curve generated by a curve rolling on another curve.

The cycloid through the origin, generated by a circle of radius $r$, consists of the points $(x, y)$, with

$$x(t) = r(t - \sin t), \quad y(t) = r(1 - \cos t)$$

where $t$ is a real parameter, corresponding to the angle through which the rolling circle has rotated, measured in radians.

For given $t$, the circle’s center lies at $x = rt, y = r$.

The cycloid satisfies the differential equation:

$$\left( \frac{dy}{dx} \right)^2 = \frac{2r}{y} - 1.$$

Johann Bernoulli stated the problem in Acta Eruditorum and, although knowing how to solve it himself, he challenged others to solve it.

**Johann Bernoulli’s solution** divides the plane into strips and he assumes that the particle follows a straight line in each strip. The path is then piecewise linear.

The problem is to determine the angle of the straight line segment in each strip and to do this he appeals to Fermat’s principle, namely that light always follows the shortest possible time of travel.

If $v$ is the velocity in one strip at angle $\alpha$ to the vertical and $u$ in the velocity in the next strip at angle $\beta$ to the vertical then, according to the usual sine law

$$\frac{v}{\sin \alpha} = \frac{u}{\sin \beta}$$

In the limit, as the strips become infinitely thin, the line segments tend to a curve where at each point the angle the line segment made with the vertical becomes the angle the tangent to the curve makes
with the vertical. If \( v \) is the velocity at \((x, y)\) and \( \alpha \) is the angle the tangent makes with the vertical then the curve satisfies \( \frac{v}{\sin \alpha} = \text{constant} \).

Bernoulli proved that the motion of the particle satisfy:

\[
\left( \frac{dy}{dx} \right)^2 = \frac{2r}{y} - 1
\]

(differential equation of cycloid).

Johann Bernoulli had posed certain \textbf{geodesic problems} to Euler which, like the brachistochrone problem, were of this type. Here the problem was \textbf{to find curves of minimum length where the curves were constrained to lie on a given surface}. 